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Multi-instanton calculus versus exact results in $N = 2$ supersymmetric QCD

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Abstract

Microscopic tests of the exact results are performed in $N = 2$ supersymmetric $SU(2)$ QCD. We present the complete construction of the multi-instanton in $N = 2$ supersymmetric QCD. All the defining equations of the super instanton are reduced to the algebraic equations. Using this result, we calculate the two-instanton contribution \mathcal{F}_2 to the prepotential \mathcal{F} for the arbitrary N_f theories. For $N_f = 0, 1, 2$, instanton calculus agrees with the prediction of the exact results, however, for $N_f = 3, 4$, we find discrepancies between them. We propose improved curves of the exact results for the massive $N_f = 3$ and massless $N_f = 4$ theories.

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1 Introduction

In the last few years, much progress has been made in the study of the strongly coupled supersymmetric gauge theories. Under the holomorphy and the duality, the low energy effective actions of $N = 2$ supersymmetric Yang-Mills theory and supersymmetric QCD in the Coulomb phase are determined exactly for $SU(2)$ gauge group[1] and later for larger gauge groups[2]-[6]. These low energy effective theories reveal the interesting results like the monopole condensation[1] and new supersymmetric conformal field theories[7, 8, 9] and so on.

The exact results predict the non-perturbative corrections from instanton. In supersymmetric theories, the dependence on the coupling constant of the instanton corrections is completely fixed by the holomorphy and symmetries. Thus it is enough to calculate in the case where the coupling constant is almost zero. In this limit, the saddle point approximation, namely the instanton calculus becomes exact. Furthermore, the instanton calculus in supersymmetric theories is safe from the infrared divergence, while the instanton calculus in the ordinary QCD is plagued by it [10, 11, 12]. Therefore, the instanton calculus gives a reliable non-trivial test of the exact results. For $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory, various checks of the exact result have been performed [13, 14, 15, 16]. And in [17] the one-instanton calculus was extended to $N = 2$ supersymmetric $SU(N_c)$ Yang-Mills theories. All the microscopic calculi above agree with the exact results.

The situation was changed in $N = 2$ supersymmetric QCD. The instanton calculus in $N = 2$ supersymmetric $SU(2)$ QCD was performed recently by two independent groups [18, 19, 20]. A discrepancy between the instanton calculus and the exact result was first observed in $N = 2$ supersymmetric $SU(2)$ QCD with $N_f = 3$ flavors [19]. Since the exact result for $N_f = 3$ is derived uniquely from the exact result for the massive $N_f = 4$ theory by the decoupling argument, it also gives a discrepancy in the massive $N_f = 4$ theory. Soon after that, for the massless $N_f = 4$ theory, another discrepancies were found in the effective coupling[20] and the moduli parameter u [19].

The aim of the present paper is to give a systematic derivation of the results presented in our short letter[19] and detail comparisons with the exact results. In $N = 2$ supersymmetric $SU(2)$ QCD, there is a parity symmetry between hypermultiplets, and then only contributions from even number of instanton exist [1]. Thus the instanton corrections start from the two-instanton sector. The construction of the supersymmetric multi-instanton in $N = 2$ supersymmetric $SU(2)$ QCD was partially presented in [19]. In this paper, we present the complete construction of the supersymmetric multi-instanton in $N = 2$ supersymmetric $SU(2)$ QCD and derive the

supersymmetric instanton action. Another approach to the related topics was developed in [20]. We also give some new results. The main new results are the extension of the two-instanton calculus to arbitrary N_f theories and a proposal of improved curves for the massive $N_f = 3$ and the massless $N_f = 4$ theories.

The outline of the paper is the following. In section 2, we briefly review the exact results. In section 3 and section 4, we construct the supersymmetric multi-instanton in $N = 2$ supersymmetric $SU(2)$ QCD. In section 3, we derive the defining equation of the supersymmetric instanton and the leading-order Lagrangian. In section 4 we extend the method used in $N = 2$ supersymmetric Yang-Mills theory[14] to $N = 2$ supersymmetric QCD and find the supersymmetric instanton action. The construction of the supersymmetric instanton is generic and not restricted in the two-instanton sector. In section 5, we perform the two-instanton calculus in $N = 2$ supersymmetric $SU(2)$ QCD with arbitrary flavors. In section 6, we compare the instanton calculus and the exact result. It is found that there exist discrepancies between them. In section 7, we summarize our conclusion and discuss the improvements of the curves of the exact result. In appendix A, B and C we give our conventions, the Lagrangian of $N = 2$ supersymmetric QCD and the supersymmetric transformation respectively. In appendix D, the regularization scheme of the instanton calculus is discussed.

2 Brief review of the exact results

In this section, we briefly review the exact results derived by Seiberg and Witten [1]. We consider the Coulomb phase of $N = 2$ supersymmetric $SU(2)$ QCD. In the generic point of the Coulomb branch, the gauge symmetry breaks spontaneously to $U(1)$ and all hypermultiplets are massive, and therefore the light degree of freedom which survives in the low energy effective theory is the $U(1)$ vector multiplet. The holomorphy determines the low energy effective Lagrangian for the massless vector multiplet except for the prepotential $\mathcal{F}(A)$;

$$\mathcal{L}_{eff} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A} \bar{A} + \int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^\alpha W_\alpha \right], \quad (2.1)$$

where A and W_α are the $N = 1$ chiral superfields in the $N = 2$ $U(1)$ vector multiplet. In the semiclassical region, the prepotential are expanded by,

$$\mathcal{F}(a) = \frac{ia^2}{4\pi} \left\{ (4 - N_f) \ln \left(\frac{a^2}{\tilde{\Lambda}_{N_f}^2} \right) + \sum_{k=0}^{\infty} \mathcal{F}_k(N_f) \left(\frac{\tilde{\Lambda}_{N_f}}{a} \right)^{(4-N_f)k} \right\}, \quad (2.2)$$

where we define $a = \langle A \rangle / 2$. The first term is the one-loop correction and the others are instanton corrections. The coefficients \mathcal{F}_{2n+1} vanish for $N_f > 1$ by the parity symmetry.

Seiberg and Witten derive the exact form of the prepotential, based on the physical conjecture: the duality and the physical interpretation to singularities of the moduli space, which is parameterized by a gauge invariant parameter $u = \langle \text{tr} \phi^2 \rangle$. They introduce $a_D = \partial \mathcal{F} / \partial a$, which is related by $N = 2$ supersymmetry to the dual photon. The pair (a_D, a) is a holomorphic section of an $SL(2, Z)$ bundle over the punctured complex u -plane. The exact results are given as the period on the torus of the holomorphic differential. For the massless theories, the tori are given by the elliptic curves,

$$N_f = 0 \quad : \quad y^2 = x^2(x - u) + \frac{1}{4} \tilde{\Lambda}_0^4 x, \quad (2.3)$$

$$N_f = 1, 2, 3 \quad : \quad y^2 = x^2(x - u) - \frac{1}{64} \tilde{\Lambda}_{N_f}^{2(4-N_f)} (x - u)^{N_f-1}, \quad (2.4)$$

and their results are given by,

$$\frac{da(u)}{du} = \frac{\sqrt{2}}{8\pi} \oint_{\alpha} \frac{dx}{y}, \quad (2.5)$$

$$\frac{da_D(u)}{du} = \frac{\sqrt{2}}{8\pi} \oint_{\beta} \frac{dx}{y}. \quad (2.6)$$

By calculating the inverse function of $a(u)$, we obtain the moduli $u(a)$. Inserting $u(a)$ to $a_D(u)$, we get the exact form of the prepotential. The exact results predict all the series of multi-instanton corrections to the low energy effective Lagrangian and the moduli u . We obtain lower order coefficients,

N_f	\mathcal{F}_1	\mathcal{F}_2	
0	-2^{-4}	$-5 \cdot 2^{-13}$	
1	0	$3 \cdot 2^{-12}$	
2	0	-2^{-11}	
3	0	-2^{-10}	

(2.7)

The coefficients of the higher order corrections are obtained systematically by Picard-Fuchs equation [21]. There is the following relation between the moduli and prepotential [22, 23, 24],

$$\begin{aligned} u(a) &= \frac{8\pi i}{4 - N_f} \left(\mathcal{F}(a) - \frac{1}{2} a \partial_a \mathcal{F}(a) \right) \\ &= 2a^2 \left\{ 1 - \frac{1}{2} \sum_{k=1}^{\infty} k \mathcal{F}_k(N_f) \left(\frac{\tilde{\Lambda}_{N_f}}{a} \right)^{(4-N_f)k} \right\}. \end{aligned} \quad (2.8)$$

In the case when $N_f=4$, Seiberg and Witten assert that the quantum theory has exact scale invariance, and that both the moduli u and prepotential \mathcal{F} receive no quantum correction,

$$u = 2a^2, \quad \mathcal{F} = \frac{1}{2}\tau a^2, \quad (2.9)$$

where $\tau = \theta/\pi + 8\pi i/g^2$ is the classical coupling constant.

When the hypermultiplets have non-vanishing bare masses, the torus is deformed by the bare masses. According to the decoupling relation: $m_{N_f}^2 \tilde{\Lambda}_{N_f}^{8-2N_f} = \tilde{\Lambda}_{N_f-1}^{8-2(N_f-1)}$, the elliptic curves corresponding to different flavors are related each other. For example, the family of the elliptic curves for the massive $N_f = 3$ theory is given by,

$$\begin{aligned} y^2 = & x^2(x-u) - \frac{1}{64}\tilde{\Lambda}_3^2(x-u)^2 - \frac{1}{64}(m_1^2 + m_2^2 + m_3^2)\tilde{\Lambda}_3^2(x-u) \\ & + \frac{1}{4}m_1m_2m_3\tilde{\Lambda}_3x - \frac{1}{64}(m_1^2m_2^2 + m_2^2m_3^2 + m_1^2m_3^2)\tilde{\Lambda}_3^2. \end{aligned} \quad (2.10)$$

By considering the decoupling limit and the massless limit, Eq.(2.3) and (2.4) are easily derived from Eq.(2.10). Since there is no discrete symmetry in the moduli space for massless $N_f = 3$, u is not completely determined by studying only the massless $N_f = 3$ theory. The freedom of adding a constant to u remains. We will find that this constant is determined to be zero by the decoupling argument of the massive $N_f = 4$ theory. The massive $N_f = 4$ curve is given by,

$$\begin{aligned} y^2 = & (x^2 - c_2^2u^2)(x - c_1u) - c_2^2(x - c_1u)^2 \sum_i m_i^2 - c_2^2(c_1^2 - c_2^2)(x - c_1u) \sum_{i>j} m_i^2 m_j^2 \\ & + 2c_2(c_1^2 - c_2^2)(c_1x - c_2^2u)m_1m_2m_3m_4 - c_2^2(c_1^2 - c_2^2)^2 \sum_{i>j>k} m_i^2 m_j^2 m_k^2, \end{aligned} \quad (2.11)$$

where $c_1 = \frac{3}{2}e_1$ and $c_2 = \frac{1}{2}(e_3 - e_2)$ and e_i are the roots of the cubic polynomial: $4x^3 - g_2(\tau)x - g_3(\tau)$. Here g_2 and g_3 are defined by $g_2 = 60\pi^{-4}G_4(\tau)$, $g_3 = 140\pi^{-6}G_6(\tau)$ and G_4, G_6 are the Eisenstein series:

$$G_4(\tau) = \sum_{m,n \in \mathbb{Z} \neq 0} \frac{1}{(m\tau + n)^4}, \quad G_6(\tau) = \sum_{m,n \in \mathbb{Z} \neq 0} \frac{1}{(m\tau + n)^6}. \quad (2.12)$$

The roots e_i obey the following equations,

$$e_1 + e_2 + e_3 = 0, \quad (2.13)$$

$$e_1 - e_2 = \theta_3^4(0, \tau), \quad (2.14)$$

$$e_3 - e_2 = \theta_1^4(0, \tau), \quad (2.15)$$

$$e_1 - e_3 = \theta_2^4(0, \tau), \quad (2.16)$$

where θ_i are the θ functions,

$$\theta_1(0, \tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2/2}, \quad (2.17)$$

$$\theta_2(0, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}, \quad (2.18)$$

$$\theta_3(0, \tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2}. \quad (2.19)$$

We define q by $q = \exp(2\pi i\tau)$. In the classical limit: $\tau \rightarrow i\infty$,

$$e_1 = \frac{2}{3} + 16q + O(q^2), \quad (2.20)$$

$$e_2 = -\frac{1}{3} - 8q^{1/2} + O(q), \quad (2.21)$$

$$e_3 = -\frac{1}{3} + 8q^{1/2} + O(q). \quad (2.22)$$

By taking the decoupling limit: $\tau \rightarrow i\infty$, $m_4 \rightarrow \infty$ with m_1, m_2, m_3 and $\Lambda_3 = 64q^{1/2}m_4$ fixed, the massive $N_f = 3$ curve (2.10) is derived uniquely and therefore the moduli u for $N_f = 3$ is determined without the freedom of adding a constant.

3 Defining equations of the supersymmetric instanton

In the following two sections, we will construct the supersymmetric multi-instanton in the Coulomb phase of $N = 2$ supersymmetric QCD. In the Coulomb phase, the adjoint scalar and the gauge boson become massive, therefore the scale invariance is broken. It is well-known that when the scale invariance is broken, instanton ceases to exist as a solution of the equation of motion except for the zero-radius one. In the formal manner, we must extend the equation of motion in order to incorporate the instanton effects [25, 26]. However, when the coupling constant is weak enough, the structure of the dominating configuration of the path integral does not depend on the details of the extension of the equation. In the weak coupling theory, the small size configuration dominates the path integral, since the size of the dominating configuration ρ is given by $\rho \sim g/M$, where g is the coupling constant and M is the Higgs mass. Because the scale invariance is effectively restored for the small size configuration, the dominating configuration satisfies the classical equation in the leading order of g except for the large-range behavior, which is not relevant in the following calculation. In supersymmetric theories, the coupling dependence of the instanton contribution is completely fixed, then it is sufficient to consider the case in which the coupling constant is almost zero. Therefore, we will solve the equation of motion in the leading order.

In non-supersymmetric theory, the leading-order equation was already given by 't Hooft [27]. To extend to the supersymmetric theory, we must take into account the source term given by the fermionic zero modes. For example, the leading equation of the $N = 2$ vector multiplet becomes

$$\begin{aligned} F_{\mu\nu} &= -\tilde{F}_{\mu\nu}, \\ \bar{D}\lambda &= 0, \quad \bar{D}\psi = 0, \\ D^2\phi - \sqrt{2}ig[\lambda, \psi] &= 0. \end{aligned} \tag{3.1}$$

Comparing to the non-supersymmetric theory, a difference appears in the equation of the scalar field, which is given by $D^2\phi = 0$ in non-supersymmetric theory. To clarify this, we estimate the coupling constant dependence of the source term. The coupling constant dependence of instanton solution is given by $A_\mu \sim O(g^{-1})$ and because of $N = 1$ supersymmetry, we set the normalization of the fermionic zero mode so that $\lambda \sim O(g^{-1})$ ¹. Again because of $N = 1$ supersymmetry, ϕ and ψ have the same coupling constant dependence, therefore, the source term of the equation of ϕ has the same order as $D^2\phi$. This is the reason why the 't Hooft equation must be modified in the supersymmetric theory. The coupling constant dependence of the ϕ and ψ is determined by the boundary condition of the ϕ . In the Coulomb branch, ϕ must satisfy the boundary condition $\phi \rightarrow \langle\phi\rangle$ at $x \rightarrow \infty$. The vacuum expectation value $\langle\phi\rangle$ does not depend on the coupling constant, then $\phi, \psi \sim O(1)$ ².

In the similar way, we can derive all the leading-order equation. In $N = 2$ supersymmetric QCD, there appear N_f hypermultiplets. The $N = 1$ fundamental chiral multiplet in the $N = 2$ hypermultiplet is characterized by the following equations:

$$\bar{D}q = 0, \quad \bar{D}\tilde{q} = 0, \tag{3.2}$$

$$D^2Q - \sqrt{2}ig\lambda q = 0, \quad D^2\tilde{Q} + \sqrt{2}ig\tilde{q}\lambda = 0. \tag{3.3}$$

As well as the vector multiplet, because of the $N = 1$ supersymmetry, a source term appears in (3.3). As will be shown below, if we demand that the kinetic term of q and \tilde{q} is the same order as that of λ , the normalization of q and \tilde{q} is determined uniquely: $q, \tilde{q} \sim O(g^{-1/2})$ and $Q, \tilde{Q} \sim O(g^{-1/2})$. By the $SU(2)_R$ symmetry

$$\lambda \rightarrow \psi, \quad Q \rightarrow \tilde{Q}^\dagger, \quad \tilde{Q} \rightarrow -Q^\dagger, \tag{3.4}$$

¹ $\bar{\lambda}$ is not generated by the super transformation of A_μ ; $\delta\bar{\lambda} \propto \sigma_{\mu\nu}F_{\mu\nu} = 0$

²We have respected only $N = 1$ supersymmetry for simplicity. Even if we take into account $N = 2$ supersymmetry, the conclusion is not changed.

the following equation of the fundamental anti-chiral multiplet is derived,

$$D^2 Q^\dagger - \sqrt{2}ig\tilde{q}\psi = 0, \quad D^2 \tilde{Q}^\dagger - \sqrt{2}ig\psi q = 0. \quad (3.5)$$

From this equation, we find that $Q^\dagger, \tilde{Q}^\dagger, \bar{q}, \tilde{\bar{q}} \sim O(g^{1/2})$.

In $N = 2$ supersymmetric QCD, the leading-order equation of the adjoint anti-scalar field is also modified,

$$D^2 \phi^{\dagger a} - \sqrt{2}ig\tilde{q}T^a q = 0. \quad (3.6)$$

The source term comes from the superpotential $-i\sqrt{2}g\tilde{Q}\Phi Q$. As well as ϕ , the boundary condition determines the order of ϕ^\dagger : $\phi^\dagger \sim O(1)$. Thus the order of super partner $\bar{\psi}$ becomes $\bar{\psi} \sim O(1)$. Note that the source term in (3.6) is the same order as $D^2 \phi^{\dagger a}$.

Since the equations of the remaining fields are not needed in the following calculation, we do not write them down explicitly. By examining the classical equation of motion carefully, we obtain the order of the remaining fields;

$$\begin{aligned} \bar{\lambda} &\sim O(g), & D &\sim O(g), \\ F_Q, F_{\tilde{Q}} &\sim O(g^{3/2}), & F_Q^\dagger, F_{\tilde{Q}}^\dagger &\sim O(g^{1/2}), \\ F_\phi &\sim O(g^2), & F_\phi^\dagger &\sim O(1). \end{aligned} \quad (3.7)$$

From the order of the fields obtained above, we find the leading-order parts of the Lagrangian \mathcal{L}_0 ,

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\text{tr}(F_{\mu\nu}F_{\mu\nu}) \\ &+ \text{tr} \left\{ -2i\bar{\lambda}\not{D}\lambda - 2i\bar{\psi}\not{D}\psi + 2(D_\mu\phi)^\dagger D_\mu\phi + 2\sqrt{2}ig\lambda[\psi, \phi^\dagger] \right\} \\ &+ (D_\mu Q)^\dagger D_\mu Q + D_\mu \tilde{Q}(D_\mu \tilde{Q})^\dagger - i\bar{q}\not{D}q - i\tilde{\bar{q}}\not{D}\tilde{q} \\ &+ \sqrt{2}ig \left(\tilde{q}\phi q + Q^\dagger \lambda q - \tilde{q}\lambda \tilde{Q}^\dagger + \tilde{q}\psi Q + \tilde{Q}\psi q \right). \end{aligned} \quad (3.8)$$

We have neglected $O(g^2)$ terms in the original Lagrangian. The first term of the leading-order Lagrangian is $O(g^{-2})$ and the remaining terms are $O(1)$. It can be easily seen that the kinetic terms of q and \tilde{q} is the same order as that of λ .

As a consistency test, we examine the Euler-Lagrangian equations of the leading-order Lagrangian \mathcal{L}_0 . Except for the gauge field, all the leading-order equations are derived from \mathcal{L}_0 . For the gauge field, the leading-order equation $D_\mu F_{\mu\nu} = 0$ is not derived from \mathcal{L}_0 , but

there appear source terms. However, since it can be shown easily that these source terms give only $O(g^2)$ corrections in the instanton action, we can neglect these terms in the defining equation of $F_{\mu\nu}$. We dub the dominating configuration defined by Eq.(3.1), (3.2), (3.3), (3.5) and (3.6) as supersymmetric instanton in $N = 2$ supersymmetric QCD.

4 Solutions of the defining equations

In this section, we solve the defining equations and give the supersymmetric instanton explicitly. We extend the result in $N = 2$ supersymmetric Yang-Mills theory[14] to $N = 2$ supersymmetric QCD. As we will show below, all the defining equations reduce to the algebraic equations. In the following, we omit the coupling constant g for simplicity.

4.1 vector multiplet

4.1.1 spin 1

In this paper, we consider the gauge field A_μ satisfying the anti-self-dual condition:

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}. \quad (4.1)$$

Anti-self-dual solutions of arbitrary number of instanton are constructed by the ADHM construction [28, 29]. We will treat the k -instanton in this section. To construct the solution, we introduce a $(k+1) \times k$ matrix $M(x)$ made up of quaternions³. The matrix $M(x)$ is chosen to be linear in x ,

$$M(x) = B - Cx, \quad (4.2)$$

where B is constant $(k+1) \times k$ quaternionic matrices of rank k and x is the quaternion. We denote the elements of B by a k -dimensional row vector ω and a $k \times k$ matrix \hat{a} ,

$$B = \begin{pmatrix} \omega_1 & \cdots & \omega_k \\ & \hat{a} & \end{pmatrix}. \quad (4.3)$$

The $(k+1) \times k$ matrix C is chosen to the following canonical form,

$$C = \begin{pmatrix} 0 & \cdots & 0 \\ & \hat{C} & \end{pmatrix}, \quad \hat{C}_{i,j} = \delta_{i,j}, \quad i, j = 1, \dots, k. \quad (4.4)$$

³See appendix A. The matrix representation is given there.

Furthermore,

$$R(x) = M^\dagger(x)M(x) \quad (4.5)$$

is assumed to be a real, invertible $k \times k$ matrix: $\bar{R}_{i,j} = R_{i,j}$. From this we obtain $C^T M_\mu = M_\mu^T C$ which is useful in the calculation. As we will see soon, the reality condition of R is equivalent to the anti-self-duality of $F_{\mu\nu}$.

The gauge field A_μ is given by a quaternionic $(k+1)$ -dimensional column vector $N(x)$:

$$A_\mu^{\dot{r}\dot{s}}(x) = iN^{\dagger\dot{r}r}(x)\partial_\mu N(x)_{r\dot{s}}. \quad (4.6)$$

$N(x)$ is given by the solution of the following algebraic equation:

$$N^\dagger(x)M(x) = 0, \quad N^\dagger(x)N(x) = 1. \quad (4.7)$$

The first equation gives k quaternionic conditions on the $k+1$ elements in $N(x)$. The second equation determines the normalization of $N(x)$. Thus there remains an ambiguity of $N(x)$; $N(x) \rightarrow N(x)u(x)$, where $u(x)$ is a quaternion of unit length, $u^\dagger(x)u(x) = 1$. In the matrix representation of quaternion, $u(x)$ is a unitary matrix. This is a gauge symmetry. Applying an appropriate gauge transformation, we set the boundary condition of the 0-th component of $N(x)$ as $N_0(x) \rightarrow 1$ at $x \rightarrow \infty$. This is the multi-instanton version of the singular gauge condition. In the singular gauge, N is given by,

$$N = \begin{pmatrix} N_0 \\ N_{0+i} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \omega_i R_{i,j}^{-1} \bar{\omega}_j} \\ -M_{0+i,j} R_{j,h}^{-1} \bar{\omega}_h N_0 / |N_0|^2 \end{pmatrix}, \quad (4.8)$$

where $i = 1, \dots, k$. In the singular gauge, the 0-th component plays a special role. To distinguish the 0-th component from others clearly, we use the index $0+i$ to denote the other components. It is easily seen that $A_\mu \rightarrow O(1/x^3)$ at $x \rightarrow \infty$.

The following formula is useful in the multi-instanton calculus:

$$I - N(x)N^\dagger(x) = M(x)R^{-1}(x)M^\dagger(x), \quad (4.9)$$

where R^{-1} is the $k \times k$ inverse matrix to R and I is the $k \times k$ unit matrix. This formula is derived from an identity $(I - NN^\dagger - MR^{-1}M^\dagger)M = 0$. From this identity, we find that $I - NN^\dagger - MR^{-1}M^\dagger \propto N^\dagger$. Since it can be shown that the proportional constant becomes zero if we multiply N to the right, we get the formula (4.9).

Now we show the anti-self-duality of $F_{\mu\nu}$. The field strength of the gauge field (4.6) is

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ &= i\partial_\mu(N^\dagger\partial_\nu N) + i(N^\dagger\partial_\mu N)(N^\dagger\partial_\nu N) - (\mu \leftrightarrow \nu) \\ &= i\partial_\mu N^\dagger\{I - NN^\dagger\}\partial_\nu N - (\mu \leftrightarrow \nu). \end{aligned} \quad (4.10)$$

Using the formula (4.9), we find

$$\begin{aligned} &= i(\partial_\mu N^\dagger)MR^{-1}M^\dagger(\partial_\nu N) - (\mu \leftrightarrow \nu) \\ &= iN^\dagger(\partial_\mu M)R^{-1}(\partial_\nu M^\dagger)N - (\mu \leftrightarrow \nu) \\ &= iN^\dagger C\sigma_\mu R^{-1}\bar{\sigma}_\nu C^T N - (\mu \leftrightarrow \nu). \end{aligned} \quad (4.11)$$

Since R^{-1} is real, it commute with σ_μ , then we obtain finally,

$$\begin{aligned} F_{\mu\nu} &= iN^\dagger CR^{-1}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)C^T N \\ &= -4N^\dagger CR^{-1}\bar{\sigma}_{\mu\nu}C^T N. \end{aligned} \quad (4.12)$$

Because of the anti-self-duality of $\bar{\sigma}_{\mu\nu}$, it is proved $F_{\mu\nu}$ satisfies the anti-self-dual equation.

4.1.2 spin 1/2

The adjoint fermions are given by the zero modes:

$$\bar{D}\lambda = 0, \quad \bar{D}\psi = 0. \quad (4.13)$$

The solutions of these equations have following forms [30],

$$\begin{aligned} \lambda_{\alpha\dot{s}}^\dagger &= N^{\dagger\dot{r}r} \left\{ \mathcal{M}_r R^{-1} C^T \delta_\alpha^s + \epsilon_{r\alpha} C R^{-1} (\mathcal{M}^T)^s \right\} N_{s\dot{s}}, \\ \psi_{\alpha\dot{s}}^\dagger &= N^{\dagger\dot{r}r} \left\{ \mathcal{N}_r R^{-1} C^T \delta_\alpha^s + \epsilon_{r\alpha} C R^{-1} (\mathcal{N}^T)^s \right\} N_{s\dot{s}}. \end{aligned} \quad (4.14)$$

Here \mathcal{M}_r and \mathcal{N}_r are constant $(k+1) \times k$ grassmannian matrices. These forms of the zero modes are anticipated by the supersymmetry since there exist zero modes corresponding to the supersymmetry in the super partner of the gauge field. If we apply the supersymmetric transformation ξ to λ , we obtain

$$\delta\lambda = -\bar{\sigma}_{\mu\nu}\xi F_{\mu\nu} = N^{\dagger\dot{r}r} \left\{ 4\xi_r C R^{-1} C^T \delta_\alpha^s + \epsilon_{r\alpha} C R^{-1} 4\xi^s C^T \right\} N_{s\dot{s}}. \quad (4.15)$$

The last equation is the same as the equation (4.14) if we replace \mathcal{M}_r with $4\xi_r C$. To hold (4.13), \mathcal{M} and \mathcal{N} must satisfy the algebraic equations. Using the formula (4.9), it can be found

$$\bar{\sigma}_\mu^{\dot{\alpha}\alpha}(D_\mu\lambda_\alpha)_{\dot{s}}^\dagger = -2iN^{\dagger\dot{r}\alpha}CR^{-1}\left\{M^{\dagger\alpha s}\mathcal{M}_s + (\mathcal{M}^T)^s M_{s\dot{q}}\epsilon^{\dot{\alpha}\dot{q}}\right\}R^{-1}C^TN_{\alpha\dot{s}}. \quad (4.16)$$

Therefore we obtain

$$M^{\dagger\dot{\alpha}s}\mathcal{M}_s + (\mathcal{M}^T)^s M_{s\dot{q}}\epsilon^{\dot{\alpha}\dot{q}} = 0. \quad (4.17)$$

In the similar way, we find a constraint equation of \mathcal{N} .

$$M^{\dagger\dot{\alpha}s}\mathcal{N}_s + (\mathcal{N}^T)^s M_{s\dot{q}}\epsilon^{\dot{\alpha}\dot{q}} = 0. \quad (4.18)$$

4.1.3 spin 0

The adjoint scalar field satisfies the following equation,

$$D^2\phi - \sqrt{2}i[\lambda, \psi] = 0. \quad (4.19)$$

Following [14], we first consider the equation without the source term,

$$D^2\phi_0 = 0, \quad (4.20)$$

which obeys the boundary condition as $\phi \rightarrow \langle\phi\rangle$ at $x \rightarrow \infty$. The solution of this equation has the following form:

$$\phi_{0\dot{s}}^{\dot{r}} = -iN^{\dagger\dot{r}r}A_r^s N_{s\dot{s}}, \quad (4.21)$$

where A_r^s is the following $(k+1) \times k$ matrix:

$$A_r^s = \begin{pmatrix} A_{0,0}^s & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \hat{A}\delta_r^s & & \\ 0 & & & \end{pmatrix}, \quad A_{0,0} = i\langle\phi\rangle, \quad \hat{A}^T = -\hat{A}. \quad (4.22)$$

Since we take the singular gauge: $N_0 \rightarrow 1$, $N_{0+i} \rightarrow 0$ at $x \rightarrow \infty$, ϕ_0 goes to $\langle\phi\rangle$ at $x \rightarrow \infty$. Substituting this to the left-hand side of (4.20), we obtain

$$\begin{aligned} (D^2\phi_0)_{\dot{s}}^{\dot{r}} &= 4iN^{\dagger\dot{r}r}CR^{-1}C^T A_r^s N_{s\dot{s}} + 4iN^{\dagger\dot{r}r}A_r^s CR^{-1}C^T N_{s\dot{s}} \\ &\quad - 4iN^{\dagger\dot{r}r}CR^{-1}M^{\dagger\dot{\alpha}\beta}A_{\beta}^{\gamma}M_{\gamma\dot{\alpha}}R^{-1}C^T N_{r\dot{s}} \\ &= 4i(N_{0+i}^{\dagger})^{\dot{r}r}R_{i,j}^{-1}\hat{A}_{j,h}(N_{0+h})_{r\dot{s}} + 4i(N_{0+i}^{\dagger})^{\dot{r}r}\hat{A}_{i,j}R_{j,h}^{-1}(N_{0+h})_{r\dot{s}} \\ &\quad - 4i(N_{0+i}^{\dagger})^{\dot{r}r}R_{i,j}^{-1}\text{tr}\left(M_{j,0+m}^{\dagger}\hat{A}_{m,n}M_{n,0+l}\right)R_{l,h}^{-1}(N_{0+h})_{r\dot{s}} \\ &\quad - 4i(N_{0+i}^{\dagger})^{\dot{r}r}R_{i,j}^{-1}\text{tr}\left(M_{j,0}^{\dagger}A_{0,0}M_{0,l}\right)R_{l,h}^{-1}(N_{0+h})_{r\dot{s}} \\ &= 4i(N_{0+i}^{\dagger})^{\dot{r}r}\left[R_{i,j}^{-1}\hat{A}_{j,h} + \hat{A}_{i,j}R_{j,h}^{-1} - R_{i,j}^{-1}\text{tr}\left(M_{j,0+m}^{\dagger}\hat{A}_{m,n}M_{n,0+l}\right)R_{l,h}^{-1}\right](N_{0+h})_{r\dot{s}} \\ &\quad - 4i(N_{0+i}^{\dagger})^{\dot{r}r}R_{i,j}^{-1}\Lambda_{j,l}R_{l,h}^{-1}(N_{0+h})_{r\dot{s}}, \end{aligned} \quad (4.23)$$

where Λ is $k \times k$ anti-symmetric matrix defined by

$$\begin{aligned}
\Lambda_{j,l} &\equiv \text{tr} \left(M_{j,0}^\dagger A_{0,0} M_{0,l} \right) \\
&= \text{tr}(\bar{\omega}_j A_{0,0} \omega_l) \\
&= \frac{1}{2} \text{tr}(A_{0,0} \{ \omega_l \bar{\omega}_j + \omega_j \bar{\omega}_l \}) + \frac{1}{2} \text{tr}(A_{0,0} \{ \omega_l \bar{\omega}_j - \omega_j \bar{\omega}_l \}) \\
&= \frac{1}{2} \text{tr}(\bar{\omega}_j A_{0,0} \omega_l - \bar{\omega}_l A_{0,0} \omega_j).
\end{aligned} \tag{4.24}$$

Here we have used $\omega_l \bar{\omega}_j + \omega_j \bar{\omega}_l \propto 1$ and $\text{tr} A_{0,0} = 0$.⁴ The first term of the right-hand side of (4.23) is simplified by the following equation,

$$\begin{aligned}
&R_{i,j}^{-1} \hat{A}_{j,h} + \hat{A}_{i,j} R_{j,h}^{-1} - R_{i,j}^{-1} \text{tr} \left(M_{j,0+m}^\dagger \hat{A}_{m,n} M_{n,0+l} \right) R_{l,h}^{-1} \\
&= R_{i,j}^{-1} \text{tr} \left(\frac{1}{2} \hat{A}_{j,m} R_{m,l} + \frac{1}{2} R_{j,m} \hat{A}_{m,l} - M_{j,0+m}^\dagger \hat{A}_{m,n} M_{n,0+l} \right) R_{l,h}^{-1} \\
&= R_{i,j}^{-1} \left([\hat{a}_\mu, [\hat{a}_\mu, \hat{A}]] + \frac{1}{2} \{ \hat{A}, W \} \right)_{j,l} R_{l,h}^{-1},
\end{aligned} \tag{4.25}$$

where \hat{a}_μ is a $k \times k$ real matrix defined by $\hat{a} = -i \hat{a}_\mu \sigma_\mu$ and W is a $k \times k$ real symmetric matrix defined by

$$W_{j,l} = \frac{1}{2} \text{tr}(\bar{\omega}_j \omega_l + \bar{\omega}_l \omega_j). \tag{4.26}$$

Finally we obtain

$$(D^2 \phi_0)_{\dot{s}}^{\dot{r}} = 4i(N_{0+i}^\dagger)^{\dot{r}r} R_{i,j}^{-1} \left([\hat{a}_\mu, [\hat{a}_\mu, \hat{A}]] + \frac{1}{2} \{ \hat{A}, W \} - \Lambda \right)_{j,l} R_{l,h}^{-1} (N_{0+h})_{r\dot{s}}. \tag{4.27}$$

Therefore, \hat{A} must satisfy an algebraic equation,

$$[\hat{a}_\mu, [\hat{a}_\mu, \hat{A}]] + \frac{1}{2} \{ \hat{A}, W \} = \Lambda. \tag{4.28}$$

Now let us solve Eq.(4.19). The solution of (4.19) with the boundary condition $\phi \rightarrow 0$ at $x \rightarrow \infty$ has the following form:

$$\phi_{f\dot{s}}^{\dot{r}} = \frac{\sqrt{2}i}{4} N^{\dagger\dot{r}r} \left\{ \mathcal{N}_r R^{-1} (\mathcal{M}^T)^s - \mathcal{M}_r R^{-1} (\mathcal{N}^T)^s \right\} N_{s\dot{s}} - i N^{\dagger\dot{r}r} F N_{r\dot{s}}, \tag{4.29}$$

where F is a $(k+1) \times (k+1)$ anti-symmetric matrix:

$$F = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \hat{F} & \\ 0 & & \end{pmatrix}, \quad \hat{F}^T = -\hat{F}. \tag{4.30}$$

⁴For the definition of ω and \hat{a} , see (4.3).

The first term is anticipated by the supersymmetry,

$$\delta\phi = -\sqrt{2}i\xi\psi = \frac{\sqrt{2}i}{4}N^{\dagger rr}\left\{\mathcal{N}_rR^{-1}(4\xi^sC^T) - (4\xi_rC)R^{-1}(\mathcal{N}^T)^s\right\}N_{ss}. \quad (4.31)$$

Due to the existence of the zero modes which are not related with supersymmetry, the second term of Eq.(4.29) is necessary. By a tedious but straightforward calculation, it is found

$$\begin{aligned} (D^2\phi_f)^{\dot{r}}_{\dot{s}} - i\sqrt{2}[\lambda, \psi]^{\dot{r}}_{\dot{s}} &= 4iN^{\dagger rr}CR^{-1}C^TFN_{r\dot{s}} + 4iN^{\dagger rr}FCR^{-1}C^TN_{r\dot{s}} \\ &\quad - 4iN^{\dagger rr}CR^{-1}M^{\dagger\dot{\alpha}\beta}FM_{\beta\dot{\alpha}}R^{-1}C^TN_{r\dot{s}} \\ &\quad + \sqrt{2}iN^{\dagger rr}CR^{-1}\left\{(\mathcal{M}^T)^s\mathcal{N}_s - (\mathcal{N}^T)^s\mathcal{M}_s\right\}R^{-1}C^TN_{r\dot{s}} \\ &= 4i(N_{0+i}^{\dagger})^{\dot{r}r}\left[R_{i,j}^{-1}\hat{F}_{j,h} + \hat{F}_{i,j}R_{j,h}^{-1} - R_{i,j}^{-1}\text{tr}\left(M_{j,0+m}^{\dagger}\hat{F}_{m,n}M_{n,0+l}\right)R_{l,h}^{-1}\right](N_{0+h})_{r\dot{s}} \\ &\quad + \sqrt{2}i(N_{0+i}^{\dagger})^{\dot{r}r}R_{i,j}^{-1}\left\{(\mathcal{M}^T)^s\mathcal{N}_s - (\mathcal{N}^T)^s\mathcal{M}_s\right\}_{j,l}R_{l,h}^{-1}(N_{0+h})_{r\dot{s}}. \end{aligned} \quad (4.32)$$

The right-hand side of the above equation is the same as the right-hand side of (4.23) if the following replacement is performed:

$$\hat{F} \rightarrow \hat{A}, \quad -\frac{\sqrt{2}}{4}\left\{(\mathcal{M}^T)^s\mathcal{N}_s - (\mathcal{N}^T)^s\mathcal{M}_s\right\} \rightarrow \Lambda. \quad (4.33)$$

Thus as well as \hat{A} , the following algebraic equation is derived,

$$\left[\hat{a}_{\mu}, \left[\hat{a}_{\mu}, \hat{F}\right]\right] + \frac{1}{2}\left\{\hat{F}, W\right\} = -\frac{\sqrt{2}}{4}\left\{(\mathcal{M}^T)^s\mathcal{N}_s - (\mathcal{N}^T)^s\mathcal{M}_s\right\}. \quad (4.34)$$

To satisfy the boundary condition $\phi \rightarrow \langle\phi\rangle$ at $x \rightarrow \infty$, we add ϕ_0 to ϕ_f ,

$$\phi = \phi_0 + \phi_f. \quad (4.35)$$

This is the solution of (4.19) [14].

In $N = 2$ supersymmetric QCD, there appears a source term in the equation of the adjoint anti-scalar field,

$$D^2\phi^{\dagger a} - \sqrt{2}i\tilde{q}T^aq = 0. \quad (4.36)$$

The source term comes from the Yukawa term $\sqrt{2}i\tilde{q}\phi q$. The fundamental zero modes q, \tilde{q} will be explained in detail in the next subsection. From the completeness condition of T^a ,

$$(T^a)^{\dot{u}}_i(T^a)^{\dot{r}}_{\dot{s}} = \frac{1}{2}\delta^{\dot{u}}_{\dot{s}}\delta^{\dot{r}}_i - \frac{1}{4}\delta^{\dot{u}}_i\delta^{\dot{r}}_{\dot{s}}, \quad (4.37)$$

the above equation becomes

$$(D^2\phi^\dagger)_{\dot{s}}^{\dot{r}} - \frac{\sqrt{2}i}{4} \{2\tilde{q}_s^\alpha q_\alpha^{\dot{r}} - \delta_s^{\dot{r}} \tilde{q}_t^\alpha q_\alpha^{\dot{t}}\} = 0. \quad (4.38)$$

Substituting the fundamental zero modes given in the next subsection, the source term becomes

$$\frac{\sqrt{2}i}{4} \{2\tilde{q}_s^\alpha q_\alpha^{\dot{r}} - \delta_s^{\dot{r}} \tilde{q}_t^\alpha q_\alpha^{\dot{t}}\} = 4iN^{\dagger\dot{r}r}CR^{-1}ZR^{-1}C^TN_{r\dot{s}}, \quad (4.39)$$

where Z is a $k \times k$ grassmannian matrix given by

$$Z_{h,l} = -\frac{\sqrt{2}}{16\pi^2}(\zeta_h\tilde{\zeta}_l - \zeta_l\tilde{\zeta}_h). \quad (4.40)$$

Eq.(4.38) can be solved by the following ansatz:

$$\phi_q^\dagger = -iN^{\dagger\dot{r}r}PN_{r\dot{s}}, \quad (4.41)$$

where P is a $(k+1) \times (k+1)$ even grassmannian matrix having the following form:

$$P = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \hat{P} & \\ 0 & & \end{pmatrix}, \quad \hat{P}^T = -\hat{P}. \quad (4.42)$$

Under this ansatz, we obtain

$$\begin{aligned} & (D^2\phi_q^\dagger)_{\dot{s}}^{\dot{r}} - \frac{\sqrt{2}i}{4} \{2\tilde{q}_s^\alpha q_\alpha^{\dot{r}} - \delta_s^{\dot{r}} \tilde{q}_t^\alpha q_\alpha^{\dot{t}}\} \\ &= 4iN^{\dagger\dot{r}r}CR^{-1}C^TPN_{r\dot{s}} + 4iN^{\dagger\dot{r}r}PCR^{-1}C^TN_{r\dot{s}} \\ & \quad - 4iN^{\dagger\dot{r}r}CR^{-1}M^{\dagger\dot{\alpha}\beta}PM_{\beta\dot{\alpha}}R^{-1}C^TN_{r\dot{s}} - 4iN^{\dagger\dot{r}r}CR^{-1}ZR^{-1}C^TN_{r\dot{s}} \\ &= 4i(N_{0+i}^\dagger)^{\dot{r}r} \left[R_{i,j}^{-1}\hat{P}_{j,h} + \hat{P}_{i,j}R_{j,h}^{-1} - R_{i,j}^{-1}\text{tr} \left(M_{j,0+m}^\dagger \hat{P}_{m,n} M_{n,0+l} \right) R_{l,h}^{-1} \right] (N_{0+h})_{r\dot{s}} \\ & \quad - 4i(N_{0+i}^\dagger)^{\dot{r}r} R_{i,j}^{-1}Z_{j,l}R_{l,h}^{-1}(N_{0+h})_{r\dot{s}}. \end{aligned} \quad (4.43)$$

As well as Eq.(4.32), the right-hand side of the above equation is the same as the right-hand side of Eq.(4.23) if we replace \hat{P} and Z with \hat{A} and Λ . Therefore, from Eq.(4.38) the matrix \hat{P} must satisfy the following algebraic equation:

$$\left[\hat{a}_\mu, \left[\hat{a}_\mu, \hat{P} \right] \right] + \frac{1}{2} \{ \hat{P}, W \} = Z. \quad (4.44)$$

The above solution ϕ_q^\dagger does not satisfy the boundary condition $\phi^\dagger \rightarrow \langle \phi \rangle^\dagger$ at $x \rightarrow \infty$. To satisfy this boundary condition, we add ϕ_0^\dagger to the above solution. Thus we obtain

$$\phi^\dagger = \phi_0^\dagger + \phi_q^\dagger. \quad (4.45)$$

4.2 hypermultiplet

4.2.1 spin 1/2

As is known by the index theorem, for k -instanton there exist k zero modes in the fundamental fermion q and \tilde{q} respectively. The explicit forms of the zero modes

$$\bar{D}q = 0, \quad \bar{D}\tilde{q} = 0 \quad (4.46)$$

are given by [31]

$$q_{f\alpha}^{\dot{r}} = -\Psi^{\dagger\dot{r}r}\epsilon_{r\alpha}\zeta_f, \quad \tilde{q}_{f\dot{s}}^\alpha = -\tilde{\zeta}_f^T\epsilon^{\alpha s}\Psi_{s\dot{s}}, \quad (4.47)$$

where Ψ is a k -dimensional quaternion column vector given by

$$\Psi_{s\dot{s}} = \frac{1}{\pi}R^{-1}C^TN_{s\dot{s}}, \quad \Psi^{\dagger\dot{r}r} = \frac{1}{\pi}N^{\dagger\dot{r}r}CR^{-1}, \quad (4.48)$$

and ζ_f and $\tilde{\zeta}_f$ are k -dimensional grassmannian column vectors which exactly correspond to k zero modes in q and \tilde{q} . The index $f = 1, \dots, N_f$ is the flavor one. Ψ is normalized as

$$\int d^4x \text{tr}(\Psi_i \bar{\Psi}_j) = \delta_{i,j}, \quad (4.49)$$

which is proved by an identity

$$\partial_\mu^2 R^{-1} = -4R^{-1}C^T \text{tr}(NN^\dagger)CR^{-1}, \quad (4.50)$$

and $R_{i,j}^{-1} \rightarrow \delta_{i,j}/x^2$ at $x \rightarrow \infty$.

4.2.2 spin 0

The defining equations of the fundamental scalar fields are

$$D^2Q - \sqrt{2}i\lambda q = 0, \quad D^2\tilde{Q} + \sqrt{2}i\tilde{q}\lambda = 0, \quad (4.51)$$

$$D^2Q^\dagger - \sqrt{2}i\tilde{q}\psi = 0, \quad D^2\tilde{Q}^\dagger - \sqrt{2}i\psi q = 0. \quad (4.52)$$

The supersymmetry gives a clue to solve these equations. Under the supersymmetry ξ , Q transforms as

$$\delta Q = -\sqrt{2}i\xi q = \frac{\sqrt{2}i}{4}N^{\hat{r}r}(4\xi_r C)R^{-1}\zeta_f. \quad (4.53)$$

Thus as well as λ , we anticipate that the solution of the above equation is obtained by $4\xi_r C \rightarrow \mathcal{M}_r$. This is indeed the case and using the $SU(2)_R$ symmetry, we find all the solutions of the defining equations:

$$Q_f^{\dot{r}} = \frac{\sqrt{2}i}{4} N^{\dagger\dot{r}r} \mathcal{M}_r R^{-1} \zeta_f, \quad \tilde{Q}_{f\dot{r}} = -\frac{\sqrt{2}i}{4} \epsilon_{\dot{r}s} N^{\dagger\dot{s}r} \mathcal{M}_r R^{-1} \tilde{\zeta}_f, \quad (4.54)$$

$$Q_{f\dot{r}}^{\dagger} = \frac{\sqrt{2}i}{4} \epsilon_{\dot{r}s} N^{\dagger\dot{s}r} \mathcal{N}_r R^{-1} \tilde{\zeta}_f, \quad \tilde{Q}_f^{\dagger\dot{r}} = \frac{\sqrt{2}i}{4} N^{\dagger\dot{r}r} \mathcal{N}_r R^{-1} \zeta_f. \quad (4.55)$$

The $SU(2)_R$ symmetry is manifest in the above solutions.

4.3 supersymmetric instanton action

Let us compute the action of the supersymmetric instanton in $N = 2$ supersymmetric QCD. As was shown in section 3, the leading part of the Lagrangian is

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{2} \text{tr}(F_{\mu\nu} F_{\mu\nu}) \\ & + \text{tr} \left\{ -2i\bar{\lambda}\bar{\not{D}}\lambda - 2i\bar{\psi}\bar{\not{D}}\psi + 2(D_\mu\phi)^\dagger D_\mu\phi + 2\sqrt{2}i\lambda[\psi, \phi^\dagger] \right\} \\ & + (D_\mu Q)^\dagger D_\mu Q + D_\mu \tilde{Q} (D_\mu \tilde{Q})^\dagger - i\bar{q}\bar{\not{D}}q - i\tilde{q}\tilde{\not{D}}\tilde{q} \\ & + \sqrt{2}i \left(\tilde{q}\phi q + Q^\dagger \lambda q - \tilde{q}\lambda\tilde{Q}^\dagger + \tilde{q}\psi Q + \tilde{Q}\psi q \right). \end{aligned} \quad (4.56)$$

Substituting the solution of the defining equations for \mathcal{L}_0 and integrating it, the supersymmetric instanton action S_0 is obtained. To carry out the integration, we convert the volume integration to the surface integration by the equation of supersymmetric instanton. From the defining equations, we find

$$S_0 = \frac{8k\pi^2}{g^2} + \int d^4x \partial_\mu \left\{ \text{tr}(2\phi^\dagger D_\mu\phi) + (D_\mu Q)^\dagger Q + (D_\mu \tilde{Q})^\dagger \tilde{Q} \right\} + \int d^4x \sqrt{2}i \left(\tilde{q}\phi q + Q^\dagger \lambda q + \tilde{Q}\psi q \right). \quad (4.57)$$

To integrate the last term, we introduce the auxiliary solution \bar{q} ,

$$\bar{q}_{f\dot{r}}^{\dot{\alpha}} = \frac{1}{4\pi} \tilde{\zeta}^T R^{-1} M^{\dagger\dot{\alpha}s} \left\{ \mathcal{M}_s R^{-1} (\mathcal{N}^T)^r - \mathcal{N}_s R^{-1} (\mathcal{M}^T)^r \right\} N_{r\dot{r}} + \frac{1}{\sqrt{2}} \tilde{\zeta}^T R^{-1} M^{\dagger\dot{\alpha}r} F N_{r\dot{r}}, \quad (4.58)$$

which satisfies the equation:

$$\not{D}\bar{q} + \sqrt{2}Q^\dagger \lambda + \sqrt{2}\bar{q}\phi_f + \sqrt{2}\tilde{Q}\psi = i\sqrt{2}\tilde{\zeta}^T \hat{F}\Psi. \quad (4.59)$$

Using the auxiliary solution \bar{q} and ϕ_q^\dagger , the last term of Eq.(4.57) becomes

$$\begin{aligned}
& \sqrt{2}i(\tilde{q}\phi q + Q^\dagger \lambda q + \tilde{Q}\psi q) \\
&= \text{tr} \left\{ 2(D^2\phi^\dagger)\phi_0 \right\} - i(\not{D}\bar{q})_{\dot{r}}^\alpha q_\alpha^{\dot{r}} + \sqrt{2} \sum_{f=1}^{N_f} \tilde{\zeta}_f^T \hat{F} \Psi_{\alpha\dot{r}} q_f^{\alpha\dot{r}} \\
&= \partial_\mu \text{tr} \left\{ 2(D_\mu\phi^\dagger)\phi_0 - 2\phi^\dagger(D_\mu\phi_0) \right\} - i\partial_\mu(\bar{q}\sigma_\mu q) + \sqrt{2} \sum_{f=1}^{N_f} \tilde{\zeta}_f^T \hat{F} \text{tr}(\Psi\Psi^\dagger)\zeta_f. \quad (4.60)
\end{aligned}$$

Therefore, the leading action is

$$\begin{aligned}
S_0 &= \frac{8k\pi^2}{g^2} + \int d^4x \partial_\mu \text{tr} \left\{ 2\phi^\dagger D_\mu\phi + 2(D_\mu\phi)^\dagger\phi_0 - 2\phi^\dagger D_\mu\phi_0 \right\} \\
&\quad + \int d^4x \partial_\mu \left\{ (D_\mu Q)^\dagger Q + (D_\mu \tilde{Q})^\dagger \tilde{Q} - i\bar{q}\sigma_\mu q \right\} + \sqrt{2} \sum_{f=1}^{N_f} \tilde{\zeta}_f^T \hat{F} \zeta_f, \quad (4.61)
\end{aligned}$$

where we have used the normalization condition of Ψ . Now all the volume integrations reduce to the surface integrations. The surface integrations are evaluated by the asymptotic behavior of the supersymmetric instanton. Since at infinity M , R^{-1} and N behaves as

$$\begin{aligned}
M &\rightarrow -Cx, \quad R^{-1} \rightarrow \frac{1}{|x|^2}, \\
N_0 &\rightarrow 1 - \frac{|\omega|^2_i}{2|x|^2}, \quad N_{0+i} \rightarrow \frac{x\bar{\omega}_i}{|x|^2}, \quad (4.62)
\end{aligned}$$

we have the following asymptotic behaviors of the supersymmetric instanton:

$$\begin{aligned}
A_{\mu\dot{s}}^{\dot{r}} &\rightarrow \frac{2}{x^4} \bar{\sigma}_0^{\dot{r}r} (\omega_i)_{r\dot{\alpha}} (\sigma_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} (\bar{\omega}_i)^{\dot{\beta}s} \sigma_{0s\dot{s}} x_\nu, \\
\phi_{0\dot{s}}^{\dot{r}} &\rightarrow -i\bar{\sigma}_0^{\dot{r}r} \left\{ (A_{0,0})_r^s \left(1 - \frac{|\omega_i|^2}{x^2} \right) + \frac{1}{x^2} (\omega_i)_{rt} \hat{A}_{i,j} (\bar{\omega}_j)^{ts} \right\} \sigma_{0s\dot{s}}, \\
\phi_{f\dot{s}}^{\dot{r}} &\rightarrow -i\bar{\sigma}_0^{\dot{r}r} \left\{ -\frac{\sqrt{2}}{4x^2} \{ (\mathcal{N}_{0,i})_r (\mathcal{M}_{0,i})^s - (\mathcal{M}_{0,i})_r (\mathcal{N}_{0,i})^s \} + \frac{1}{x^2} (\omega_i)_{rt} \hat{F}_{i,j} (\bar{\omega}_j)^{ts} \right\} \sigma_{0s\dot{s}}, \\
\phi_{q\dot{s}}^{\dagger\dot{r}} &\rightarrow -i\bar{\sigma}_0^{\dot{r}r} \left\{ \frac{1}{x^2} (\omega_i)_{rt} \hat{P}_{i,j} (\bar{\omega}_j)^{ts} \right\} \sigma_{0s\dot{s}}. \quad (4.63)
\end{aligned}$$

We have omitted the asymptotic behaviors of the hypermultiplets, since they vanish faster than $O(x^{-2})$ and do not contribute the integration. Finally we obtain the following supersymmetric instanton action:

$$\begin{aligned}
S_0 &= \frac{8k\pi^2}{g^2} + S_{higgs} + S_{yukawa}, \\
S_{higgs} &= 16\pi^2 |A_{0,0}|^2 |\omega_i|^2 - 8\pi^2 \text{tr}(\bar{\omega}_i A_{0,0}^\dagger \omega_j) \hat{A}_{j,i}
\end{aligned}$$

$$\begin{aligned}
S_{yukawa} = & 2\sqrt{2}\pi^2 \text{tr} \left\{ \mathcal{N}_{0,i} A_{0,0}^\dagger \mathcal{M}_{0,i} - \mathcal{M}_{0,i} A_{0,0}^\dagger \mathcal{N}_{0,i} \right\} - 8\pi^2 \text{tr}(\bar{\omega}_i A_{0,0}^\dagger \omega_j) \hat{F}_{j,i} \\
& + 8\pi^2 \hat{P}_{i,j} \text{tr}(\bar{\omega}_j A_{0,0} \omega_i) + \sqrt{2} \sum_{f=1}^{N_f} \tilde{\zeta}_f^T \hat{F} \zeta_f,
\end{aligned} \tag{4.64}$$

where $|A_{0,0}|^2 = \text{tr}\{A_{0,0}^\dagger A_{0,0}\}/2$.

5 Instanton calculus

In this section, we will perform the instanton calculus by using the supersymmetric instanton constructed in the previous sections. We calculate the moduli $u = \langle \text{tr} \phi^2 \rangle$ as the function of $\langle \phi \rangle$. We choose $\langle \phi \rangle = a\sigma_3/2$, where a is real. In the classical limit, $u \sim a^2/2$ in this convention, which differs from a in section 2 by factor 2. Taking into account the super transformations, it is easy to find that the adjoint scalar ϕ contains the following part;

$$\phi = -\sqrt{2}i\xi\psi + \dots = \sqrt{2}i\xi\bar{\sigma}_{\mu\nu}\xi'F_{\mu\nu} + \dots, \tag{5.1}$$

where ξ, ξ' are the grassmannian collective coordinates accompanying $N = 2$ supersymmetry and \dots includes the other fermionic zero modes and ϕ_0 . Then, $\text{tr}\phi^2$ is given by,

$$\text{tr}\phi^2 = -2\text{tr} \left[(\xi\bar{\sigma}_{\mu\nu}\xi'F_{\mu\nu})^2 \right] + \dots \tag{5.2}$$

$$= -\xi^2\xi'^2\text{tr}(F_{\mu\nu}F_{\mu\nu}) + \dots. \tag{5.3}$$

Therefore supersymmetric zero modes are saturated by inserting $\text{tr}\phi^2$, and we obtain the following result by performing the integration over the center of the instanton;

$$\int d^4x_0 \int d^2\xi d^2\xi' \text{tr}\phi^2 = - \int d^4x_0 \text{tr}[F_{\mu\nu}(x-x_0)F_{\mu\nu}(x-x_0)] = -16\pi^2 k, \tag{5.4}$$

for k -instanton. The other fermionic modes are lifted by the Yukawa terms in the action and saturated by pulling down those terms from the action.

For one-instanton sector, the result is given by [13],

$$u_1 = \frac{1}{2}a^2 \cdot 2 \left(\frac{\Lambda_0}{a} \right)^4, \tag{5.5}$$

for $N_f = 0$. We denote k -instanton correction to u by u_k . For massless $N_f > 0$, u_1 vanishes due to the parity symmetry.

For $N_f > 0$, the leading-order correction comes from the two-instanton sector. First we construct the supersymmetric two-instanton, according to the previous section. In the two-instanton sector, \hat{a} is a 2×2 matrix made up of quaternions;

$$\hat{a} = \begin{pmatrix} x_0 + a_3 & a_1 \\ a_1 & x_0 - a_3 \end{pmatrix}, \quad a_1 = \frac{a_3}{4|a_3|^2} (\bar{\omega}_2 \omega_1 - \bar{\omega}_1 \omega_2), \quad (5.6)$$

which satisfies the reality condition of $R = M^\dagger M$. We define the grassmannian collective coordinates by the following;

$$\mathcal{M}_s = \begin{pmatrix} \mu_{1s} & \mu_{2s} \\ 4\xi_s + m_{3s} & m_{1s} \\ m_{1s} & 4\xi_s - m_{3s} \end{pmatrix}, \quad \mathcal{N}_s = \begin{pmatrix} \nu_{1s} & \nu_{2s} \\ 4\xi'_s + n_{3s} & n_{1s} \\ n_{1s} & 4\xi'_s - n_{3s} \end{pmatrix}, \quad (5.7)$$

$$m_1 = \frac{a_3}{2|a_3|^2} (2\bar{a}_1 m_3 + \bar{\omega}_2 \mu_1 - \bar{\omega}_1 \mu_2), \quad n_1 = \frac{a_3}{2|a_3|^2} (2\bar{a}_1 n_3 + \bar{\omega}_2 \nu_1 - \bar{\omega}_1 \nu_2), \quad (5.8)$$

which satisfy the constraint relations for \mathcal{M}_s and \mathcal{N}_s , respectively. By solving Eq.(4.28), Eq.(4.34) and Eq.(4.44), the matrix \hat{A} , \hat{F} and \hat{P} are given by

$$\hat{A} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad (5.9)$$

$$\alpha = -\frac{\omega}{H}, \quad (5.10)$$

$$\beta = -\frac{\sqrt{2}}{4H} (\mu_1 \nu_2 - \mu_2 \nu_1 + 2m_3 n_1 - 2m_1 n_3), \quad (5.11)$$

$$\gamma = \frac{\sqrt{2}}{16\pi^2} \frac{\sum_{f=1}^{N_f} \tilde{\zeta}_f \zeta_f}{H}, \quad (5.12)$$

where

$$L = |\omega_1|^2 + |\omega_2|^2, \quad H = L + 4|a_1|^2 + 4|a_3|^2, \quad (5.13)$$

$$\Omega = \omega_1 \bar{\omega}_2 - \omega_2 \bar{\omega}_1, \quad \omega = \frac{1}{2} \text{tr} (\Omega A_{0,0}),$$

and $\tilde{\zeta}_f \zeta_f = \tilde{\zeta}_f^i \zeta_{fi} = \epsilon^{ij} \tilde{\zeta}_{fj} \zeta_{fi} = \tilde{\zeta}_{f2} \zeta_{f1} - \tilde{\zeta}_{f1} \zeta_{f2}$. By using the above collective coordinates, the action of supersymmetric instanton becomes

$$\begin{aligned} S &= \frac{16\pi^2}{g^2} + S_{higgs} + S_{yukawa}, \\ S_{higgs} &= 16\pi^2 \left(L|A_{0,0}|^2 - \frac{\omega^2}{H} \right), \\ S_{yukawa} &= -4\sqrt{2}\pi^2 \left\{ \nu_k A_{0,0} \mu_k + \frac{\omega}{H} (\mu_1 \nu_2 - \mu_2 \nu_1 + 2m_3 n_1 - 2m_1 n_3) \right\} \\ &\quad + \frac{1}{2H} (\mu_1 \nu_2 - \mu_2 \nu_1 + 2m_3 n_1 - 2m_1 n_3) \sum_{f=1}^{N_f} \tilde{\zeta}_f \zeta_f + \sqrt{2} \frac{\omega}{H} \sum_{f=1}^{N_f} \tilde{\zeta}_f \zeta_f, \end{aligned} \quad (5.14)$$

where $A_{0,0} = i \langle \phi \rangle = ia\sigma^3/2$ and then $|A_{0,0}|^2 = a^2/4$. Compared with the pure Yang-Mills case[14], the last two terms in S_{yukawa} are added. Note that a biquadratic term in grassmannian variables appears in the action. This is a new feature in $N = 2$ supersymmetric QCD. The measure of the collective coordinate is given by ⁵ [32, 14],

$$C_J \int d^4x_0 d^4a_3 d^4\omega_1 d^4\omega_2 d^2\xi d^2m_3 d^2\mu_1 d^2\mu_2 d^2\xi' d^2n_3 d^2\nu_1 d^2\nu_2 \quad (5.15)$$

$$\times \prod_{f=1}^{N_f} d^2\zeta_f d^2\tilde{\zeta}_f \frac{||a_3|^2 - |a_1|^2|}{H} \exp(-S_{higgs} - S_{yukawa}),$$

$$C_J = 2^{6+2N_f} \pi^{-8} \Lambda_{N_f}^{8-2N_f}, \quad (5.16)$$

where $\Lambda_{N_f}^{8-2N_f}$ is replaced by $q = e^{2\pi i\tau}$ for $N_f = 4$.

The supersymmetric zero modes ξ and ξ' are saturated by inserting $\text{tr}\phi^2$, as we have already seen. The other fermionic modes are lifted by the Yukawa terms in the action, and integrating out those modes except $\zeta_f, \tilde{\zeta}_f$, we obtain

$$\int d^2m_3 d^2\mu_1 d^2\mu_2 d^2n_3 d^2\nu_1 d^2\nu_2 \exp(-S_{yukawa})$$

$$= - \left(\frac{16\sqrt{2}\pi^6}{|a_3|^2 H |\Omega|} \right)^2 f(y) \exp \left(-\sqrt{2} \frac{\omega}{H} \sum_{f=1}^{N_f} \tilde{\zeta}_f \zeta_f \right), \quad (5.17)$$

where

$$f(y) = \omega^2 y^2 \left\{ \left(|\Omega|^2 |A_{0,0}|^2 + \frac{L\omega^2 y}{H} \right)^2 + \frac{L^2 - |\Omega|^2}{H^2} \omega^2 y^2 (|A_{0,0}|^2 |\Omega|^2 - \omega^2) \right\}, \quad (5.18)$$

$$y = 1 - \frac{\sqrt{2}}{16\pi^2\omega} \sum_{f=1}^{N_f} \tilde{\zeta}_f \zeta_f. \quad (5.19)$$

The remaining grassmannian integrations are performed as follows;

$$\int \prod_{f=1}^{N_f} d^2\tilde{\zeta}_f d^2\zeta_f f(y) \exp \left(-\sqrt{2} \frac{\omega}{H} \sum_{g=1}^{N_f} \tilde{\zeta}_g \zeta_g \right) = \left(-\frac{1}{2} \frac{\omega^2}{H^2} \right)^{N_f} \sum_{k=0}^{2N_f} {}_{2N_f}C_k \left(\frac{H}{16\pi^2\omega^2} \right)^k \frac{\partial^k f}{\partial y^k} \Big|_{y=1}. \quad (5.20)$$

We change the integration variables from a_3, ω_1, ω_2 to H, L, Ω , and then the measure of the integration becomes,

$$\int d^4a_3 \frac{||a_3|^2 - |a_1|^2|}{|a_3|^4} = \frac{\pi^2}{2} \int_{L+2|\Omega|}^{\infty} dH, \quad (5.21)$$

$$\int d^4\omega_1 d^4\omega_2 = \frac{\pi^3}{8} \int_0^{\infty} dL \int_{|\Omega| \leq L} d^3\Omega. \quad (5.22)$$

⁵The definition of the numerical factor of the measure depends on the regularization scheme. See appendix D for further details.

With the change to a polar coordinate: $\omega = |\Omega| |A_{0,0}| \cos \theta$ and the rescaling: $\Omega' = \Omega/L$ and $H' = H/L$, the measure is given by,

$$\begin{aligned} & \frac{\pi^5}{16} \int_0^\infty dL \int_{|\Omega| \leq L} d^3\Omega \int_{L+2|\Omega|}^\infty dH \\ &= \frac{\pi^6}{8} \int_0^\infty dL L^4 \int_{-1}^1 d(\cos \theta) \int_0^1 |\Omega'|^2 d|\Omega'| \int_{1+2|\Omega'|}^\infty dH', \end{aligned} \quad (5.23)$$

and $f(y)$ becomes

$$f(y) = |A_{0,0}|^6 |\Omega'|^6 L^6 \cos^2 \theta G(y; |\Omega'|, H', \theta), \quad (5.24)$$

where

$$G(y; |\Omega'|, H', \theta) = y^2 \left\{ \left(1 + \frac{y}{H'} \cos^2 \theta \right)^2 + \frac{1 - |\Omega'|^2}{4H'^2} y^2 \sin^2 2\theta \right\}. \quad (5.25)$$

Using Eq.(5.4), (5.15), (5.17), (5.20), (5.23) and (5.24) and performing the integration of L , we obtain the two-instanton correction to u ,

$$u_2 = \frac{1}{2} a^2 \left(\frac{\Lambda_{N_f}}{a} \right)^{8-2N_f} \cdot \left(-\frac{1}{2} \right)^{N_f} I(N_f), \quad (5.26)$$

where $I(N_f)$ is defined by

$$\begin{aligned} I(N_f) &= \int_{-1}^1 d(\cos \theta) \cos^2 \theta \int_0^1 d|\Omega'| |\Omega'|^6 \int_{1+2|\Omega'|}^\infty \frac{dH'}{H'^3} \left(\frac{|\Omega'| \cos \theta}{H'} \right)^{2N_f} \sum_{k=0}^K {}_{2N_f}C_k \\ &\quad \times (5-k)! \left(1 - \frac{|\Omega'|^2 \cos^2 \theta}{H'} \right)^{k-6} \left(\frac{H'}{|\Omega'|^2 \cos^2 \theta} \right)^k \frac{\partial^k}{\partial y^k} G(y; |\Omega'|, H', \theta) \Big|_{y=1}, \end{aligned} \quad (5.27)$$

and $K = \min[4, 2N_f]$. The evaluation of $I(N_f)$ is complicated but straightforward. Finally we obtain

$$u_2 = \frac{1}{2} a^2 \times \begin{cases} \frac{5}{2} \left(\frac{\Lambda_0}{a} \right)^8 & \text{for } N_f = 0, \\ (-1)^{N_f} \frac{2N_f - 1}{2^{N_f+1} 3^{2N_f-3}} \left(\frac{\Lambda_{N_f}}{a} \right)^{8-2N_f} & \text{for } N_f \geq 1, \end{cases} \quad (5.28)$$

where $\Lambda_{N_f}^{8-2N_f}$ is replaced with q for $N_f = 4$. In the case when the hypermultiplets have non-vanishing bare masses, the mass term is added to the action: $S_{mass} = -i \sum_{f=1}^{N_f} \sum_{i=1}^2 m_f \tilde{\zeta}_{fi} \zeta_{fi}$. The instanton calculus is performed straightforwardly. Using the equation,

$$\sum_{i,j=1}^2 \tilde{\zeta}_i^i \zeta_j \tilde{\zeta}_j = 0, \quad \left(\sum_{i=1}^2 \tilde{\zeta}_i^i \zeta_i \right)^2 = \left(\sum_{i=1}^2 \tilde{\zeta}_i \zeta_i \right)^2, \quad (5.29)$$

we obtain the result;

$$u_2 = \frac{1}{2}a^2 \left(\frac{\Lambda_{N_f}}{a}\right)^{8-2N_f} \sum_{t_1=0}^1 \cdots \sum_{t_{N_f}=0}^1 \left(-\frac{1}{2}\right)^{N_f-M} I(N_f - M) a^{-2M} \prod_{g=1}^{N_f} m_g^{2t_g}, \quad (5.30)$$

where $M = \sum_{f=1}^{N_f} t_f$, and $\Lambda_{N_f}^{8-2N_f}$ is replaced by q for $N_f = 4$. When $m_{N_f} \rightarrow \infty$ and $\Lambda_{N_f} \rightarrow 0$, we obtain

$$u_2 = \frac{1}{2}a^2 \frac{m_{N_f}^2 \Lambda_{N_f}^{8-2N_f}}{a^{8-2(N_f-1)}} \cdot \left(-\frac{1}{2}\right)^{N_f-1} I(N_f - 1), \quad (5.31)$$

where we set the other masses to be zero. From Eq.(5.26) and (5.31), we find that the result for the massive theories agrees with the decoupling relation: $m_{N_f}^2 \Lambda_{N_f}^{8-2N_f} = \Lambda_{N_f-1}^{8-2(N_f-1)}$. For $N_f = 4$, the decoupling relation is given by $m_4^2 q = \Lambda_3^2$.

6 Exact results versus instanton calculus

In the previous section, we obtain the following results by the instanton calculus for the massless theories,

$$u_1 = 2a^2 \times \begin{cases} \frac{1}{2^3} \left(\frac{\Lambda_0}{a}\right)^4 & \text{for } N_f = 0, \\ 0 & \text{for } N_f \geq 1, \end{cases} \quad (6.1)$$

$$u_2 = 2a^2 \times \begin{cases} \frac{5}{2^9} \left(\frac{\Lambda_0}{a}\right)^8 & \text{for } N_f = 0, \\ (-1)^{N_f} \frac{2N_f - 1}{2^{9-N_f} 3^{2N_f-3}} \left(\frac{\Lambda_{N_f}}{a}\right)^{8-2N_f} & \text{for } N_f \geq 1, \end{cases} \quad (6.2)$$

and the decoupling relation is given by

$$m_{N_f}^2 \Lambda_{N_f}^{8-2N_f} = \Lambda_{N_f-1}^{8-2(N_f-1)}. \quad (6.3)$$

For $N_f = 4$, $\Lambda_{N_f}^{8-2N_f}$ is replaced by q . Note on the convention for a . The definition of a differs between Eq.(5.5), Eq.(5.28) and Eq.(6.1), Eq.(6.2) by factor 2.

On the other hand, from Eq.(2.7) and Eq.(2.8) the exact result predicts the moduli $u(a)$ for the massless theories,

$$u_1 = 2a^2 \times \begin{cases} \frac{1}{2^5} \left(\frac{\tilde{\Lambda}_0}{a}\right)^4 & \text{for } N_f = 0, \\ 0 & \text{for } N_f \geq 1, \end{cases} \quad (6.4)$$

$$u_2 = 2a^2 \times \begin{cases} \frac{5}{2^{13}} \left(\frac{\tilde{\Lambda}_0}{a} \right)^8 & \text{for } N_f = 0, \\ -\frac{3}{2^{12}} \left(\frac{\tilde{\Lambda}_1}{a} \right)^6 & \text{for } N_f = 1, \\ \frac{1}{2^{11}} \left(\frac{\tilde{\Lambda}_2}{a} \right)^4 & \text{for } N_f = 2, \\ \frac{1}{2^{10}} \left(\frac{\tilde{\Lambda}_3}{a} \right)^2 & \text{for } N_f = 3, \end{cases} \quad (6.5)$$

and the decoupling relation is given by

$$m_{N_f}^2 \tilde{\Lambda}_{N_f}^{8-2N_f} = \tilde{\Lambda}_{N_f-1}^{8-2(N_f-1)} \quad \text{for } 1 \leq N_f \leq 3, \quad (6.6)$$

$$\tilde{\Lambda}_3^2 = 64^2 m_4^2 q. \quad (6.7)$$

To compare the exact result and the instanton calculus, we must relate dynamical scales $\tilde{\Lambda}_{N_f}$ and Λ_{N_f} . Only when $N_f=0$, the one-instanton contribution u_1 does not vanish, and it is easily found from Eq.(6.1) and Eq.(6.4) that the instanton calculus is consistent with the exact result in the one-instanton sector, if we identify the dynamical scales as $\tilde{\Lambda}_0 = \sqrt{2}\Lambda_0$ [13]. In supersymmetric $SU(3)$ Yang-Mills theory, the instanton calculus is consistent with the instanton calculus in the one-instanton sector, when the above relation of the dynamical scales of the $SU(2)$ theory holds[17].

The two-instanton correction to $\langle u \rangle$ is given by u_2 . According to the exact result, no quantum correction to $\langle u \rangle$ exists for $N_f=4$. Using the relation $\tilde{\Lambda}_0 = \sqrt{2}\Lambda_0$ and the decoupling relations Eq.(6.3), (6.6) we obtain the relation between the dynamical scales:

$$\tilde{\Lambda}_{N_f}^{8-2N_f} = 16\Lambda_{N_f}^{8-2N_f}. \quad (6.8)$$

From Eq.(6.2), (6.5) and (6.8), we find that the microscopic instanton calculus agrees with the exact results for $N_f = 1, 2$ as well as $N_f = 0$.⁶ However we also find discrepancies between them for $N_f = 3, 4$. For $N_f = 3$, the difference of the moduli u between the instanton calculus and the exact result is a constant. For $N_f = 4$, we find the quantum correction to the moduli u , which disagrees with the assumption used by [1].

At first sight, the decoupling relations connecting the $N_f = 3$ and $N_f = 4$ theories give an inconsistency between the exact result and the instanton calculus. The relation of the

⁶Furthermore, for the massive $N_f = 1, 2$ theories, it can be shown that the instanton calculus Eq.(5.30) agrees with the exact results[33].

instanton calculus $m_4^2 q = \Lambda_3^2$ and that of the exact result $64^2 m_4^2 q = \tilde{\Lambda}_3^2$ disagree with Eq.(6.8). However, this inconsistency can be resolved by changing the regularization scheme of the instanton calculus. We will discuss this point in more detail in appendix D.

In the similar way, it can be evaluated the four-point function $\langle \bar{\lambda} \bar{\lambda} \bar{\psi} \bar{\psi} \rangle$ by the instanton calculus[13, 14, 18], and we find that the non-trivial relation Eq.(2.8) holds for $N_f = 1, 2$ as well as $N_f = 0$ [16, 14]. For $N_f = 3, 4$, this four-point function does not depend on \mathcal{F}_2 and therefore it is not useful to check the exact result. For $N_f = 4$, the finite correction to the coupling constant is calculated in [20]: $\tau_{eff} = \tau + (i/2\pi) \sum_k \mathcal{F}_k q^{k/2}$ and we find that Eq.(2.8) with $\Lambda_{N_f}^{8-2N_f}$ replaced by q holds in this case:

$$u(a) = 2a^2 \left\{ 1 - \frac{1}{2} \sum_{k=1}^{\infty} k \mathcal{F}_k q^{k/2} \right\} = 8\pi i q \frac{\partial \mathcal{F}(a)}{\partial q}, \quad (6.9)$$

where the prepotential is given by $\mathcal{F} = \tau_{eff} a^2 / 2$.

7 Conclusion

We have constructed the supersymmetric multi-instanton in $N = 2$ supersymmetric QCD and derived the supersymmetric instanton action. The instanton calculus has been performed in $N = 2$ supersymmetric QCD and we have found that the instanton calculus agrees with the Seiberg-Witten's result for the $N_f \leq 2$ theories. We have also found the discrepancies between them for $N_f = 3, 4$. These results mean that the Seiberg-Witten solution should be modified for the $N_f = 3, 4$ theories. For $N_f = 3$ the curve is determined except for a constant added to u , since there is no discrete symmetry in the u plane. We can shift u in the elliptic curve for $N_f = 3$ by the constant, so that the discrepancy for $N_f = 3$ is resolved. We obtain the correct curve for the massive $N_f = 3$ theory:

$$y^2 = x^2 \left(x - u - \frac{1}{2^4 3^3} \tilde{\Lambda}_3^2 \right) - \frac{1}{64} \tilde{\Lambda}_3^2 \left(x - u - \frac{1}{2^4 3^3} \tilde{\Lambda}_3^2 \right)^2 - \frac{1}{64} (m_1^2 + m_2^2 + m_3^2) \tilde{\Lambda}_3^2 \left(x - u - \frac{1}{2^4 3^3} \tilde{\Lambda}_3^2 \right) + \frac{1}{4} m_1 m_2 m_3 \tilde{\Lambda}_3 x - \frac{1}{64} (m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2) \tilde{\Lambda}_3^2. \quad (7.10)$$

This modification of the curve does not affect the $N_f < 3$ theories. In the decoupling limit: $\tilde{\Lambda}_3 \rightarrow 0$, $m_3 \rightarrow \infty$ with $\tilde{\Lambda}_2^4 = m_3^2 \tilde{\Lambda}_3^2$ fixed, we obtain the massive $N_f = 2$ curve, which agrees with the massive $N_f = 2$ curve derived from the original $N_f = 3$ curve.

We also need to solve the discrepancy for the massless $N_f = 4$ theory. The quantum correction does exist in this case, contrary to [1]. The prepotential is given by $\mathcal{F}(a) = \tau_{eff} a^2 / 2$

and the moduli u is given by

$$u = 8\pi i q \frac{\partial \mathcal{F}(a)}{\partial q} = 2a^2 \frac{d\tau_{eff}}{d\tau}, \quad (7.11)$$

$$a(u) = \frac{\sqrt{2u}}{2} \left(\frac{d\tau_{eff}}{d\tau} \right)^{-1/2}, \quad a_D(u) = \tau_{eff} a(u). \quad (7.12)$$

The period of the meromorphic one-form on the Seiberg-Witten curve for the massless $N_f = 4$ theory gives the above result by replacing τ and u in the elliptic curve with τ_{eff} and $(d\tau_{eff}/d\tau)^{-1}u$, respectively:

$$y^2 = \left(x^2 - c_2 (\tau_{eff})^2 u^2 \left(\frac{d\tau_{eff}}{d\tau} \right)^{-2} \right) \left(x - c_1 (\tau_{eff}) u \left(\frac{d\tau_{eff}}{d\tau} \right)^{-1} \right). \quad (7.13)$$

There remains an unsolved inconsistency between the instanton calculus and the exact result for the massive $N_f = 4$ theory. The curve for the massive $N_f = 4$ theory should be modified, such that it satisfies the desirable conditions in both $m_4 \rightarrow 0$ and ∞ limits. The curve must go to the modified $N_f = 3$ curve in the decoupling limit: $m_4 \rightarrow \infty$ and $q \rightarrow 0$ with $m_4^2 q$ fixed and go to the modified massless $N_f = 4$ curve in the massless limit: $m_4 \rightarrow 0$.

We also have calculated the instanton corrections for $N_f > 4$. Although the theory is not asymptotically free, there exist the exact results[35] and the comparison between them may give some new insights of the theory.

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Appendices

A Conventions

In this section, we summarize our conventions used in this paper. The sigma matrix is defined by

$$(\sigma_\mu)_{\alpha\dot{\beta}} = (\boldsymbol{\sigma}, i), \quad (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} = (\boldsymbol{\sigma}, -i), \quad (\text{A.14})$$

$$(\sigma_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} = \frac{1}{4i}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu), \quad (\bar{\sigma}_{\mu\nu})_\alpha^\beta = \frac{1}{4i}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu), \quad (\text{A.15})$$

where $\boldsymbol{\sigma}$ is the Pauli matrices. The invariant tensor is defined by

$$\epsilon^{12} = \epsilon_{21} = 1, \quad \epsilon_{12} = \epsilon^{21} = -1, \quad (\text{A.16})$$

and we use the same contraction rule of the spinor index as [34]. The quaternion M given by

$$\begin{aligned} M &= M_1\hat{i} + M_2\hat{j} + M_3\hat{k} + M_4, \\ \hat{i}^2 &= \hat{j}^2 = \hat{k}^2 = -1, \quad \hat{i} \times \hat{j} = \hat{k}, \end{aligned} \quad (\text{A.17})$$

has the following matrix representation:

$$M_{s\dot{s}} = -i(\sigma_\mu)_{s\dot{s}}M_\mu. \quad (\text{A.18})$$

We denote the quaternion conjugate of M as \bar{M} . Namely

$$\bar{M} = -M_1\hat{i} - M_2\hat{j} - M_3\hat{k} + M_4, \quad (\text{A.19})$$

and it has the following matrix representation:

$$\bar{M}^{\dot{r}r} = i(\bar{\sigma}_\mu)^{\dot{r}r}M_\mu. \quad (\text{A.20})$$

For a quaternion matrix M_{IJ} , we use the symbol \dagger as the transpose of the quaternion conjugate of M_{IJ} :

$$(M^\dagger)_{IJ} = \bar{M}_{JI}. \quad (\text{A.21})$$

Since the instanton mixes the color and spinor indices, it is convenient to use the dotted index for the color index. We denote the color index as

$$A_\mu^{\dot{r}}, \quad q^{\dot{r}}, \quad \tilde{q}_{\dot{s}}. \quad (\text{A.22})$$

B Lagrangian of $N = 2$ supersymmetric $SU(2)$ QCD

In Euclidean space we have the Lagrangian,

$$\begin{aligned}
\mathcal{L} = \text{tr} \bigg\{ & \frac{1}{2} F_{\mu\nu} F_{\mu\nu} - 2i\bar{\lambda}\not{D}\lambda - D^2 + 2(D_\mu\phi)^\dagger D_\mu\phi - 2i\bar{\psi}\not{D}\psi - 2F^\dagger F \\
& + 2\sqrt{2}ig \left(\lambda[\psi, \phi^\dagger] + \bar{\lambda}[\bar{\psi}, \phi] \right) - 2gD[\phi^\dagger, \phi] \bigg\} \\
& + (D_\mu Q)^\dagger D_\mu Q + D_\mu \tilde{Q} (D_\mu \tilde{Q})^\dagger - i\bar{q}\not{D}q - i\bar{\tilde{q}}\not{D}\tilde{q} - F_Q^\dagger F_Q + F_{\tilde{Q}} F_{\tilde{Q}}^\dagger \\
& + \sqrt{2}ig \left(Q^\dagger \lambda q - \bar{q}\bar{\lambda}Q - \tilde{q}\lambda\tilde{Q}^\dagger + \tilde{Q}\bar{\lambda}\tilde{q} \right) + g \left(Q^\dagger DQ - \tilde{Q}D\tilde{Q}^\dagger \right) \\
& + \sqrt{2}ig \left(\tilde{q}\phi q + \tilde{q}\psi Q + \tilde{Q}\psi q - \bar{q}\phi^\dagger \tilde{q} - Q^\dagger \bar{\psi}\tilde{q} - \bar{q}\bar{\psi}\tilde{Q}^\dagger \right) \\
& - i\sqrt{2}g \left(F_{\tilde{Q}}\phi Q + \tilde{Q}FQ + \tilde{Q}\phi F_Q - Q^\dagger \phi^\dagger F_Q^\dagger - Q^\dagger F^\dagger \tilde{Q}^\dagger - F_Q^\dagger \phi^\dagger \tilde{Q}^\dagger \right),
\end{aligned} \tag{B.23}$$

where color, flavor and spin indices are suppressed. The covariant derivatives and the field strength are defined by,

$$D_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi], \tag{B.24}$$

$$D_\mu Q = (\partial_\mu - igA_\mu)Q, \quad D_\mu \tilde{Q} = (\partial_\mu \tilde{Q} + ig\tilde{Q}A_\mu), \tag{B.25}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{B.26}$$

We define the $N = 2$ supersymmetric mass term by,

$$\mathcal{L}_m = -i \sum_{f=1}^{N_f} m_f \left(-\tilde{q}_f q_f + F_{\tilde{Q}_f} Q_f + \tilde{Q}_f F_{Q_f} \right) + h.c. \quad . \tag{B.27}$$

In this convention, (λ, ψ) and (Q, \tilde{Q}^\dagger) are $SU(2)_R$ doublets. We adopt the superpotential: $-\sqrt{2}i\tilde{Q}\Phi Q - i\sum_{f=1}^{N_f} m_f \tilde{Q}_f Q_f$, which coincides with one used in [1] by replacing $-i\tilde{Q}_f \rightarrow \tilde{Q}_f$.

C Supersymmetry

The action has $N = 2$ supersymmetry. One of the super transformations is given by,

$$\begin{aligned}
\delta\phi &= -\sqrt{2}i\xi\psi \\
\delta\phi^\dagger &= \sqrt{2}i\bar{\xi}\bar{\psi} \\
\delta\psi &= -\sqrt{2}i\xi F - \sqrt{2}\sigma_\mu\bar{\xi}D_\mu\phi \\
\delta\bar{\psi} &= -\sqrt{2}\xi\sigma_\mu D_\mu\phi^\dagger + \sqrt{2}i\bar{\xi}F^\dagger
\end{aligned}$$

$$\begin{aligned}
\delta F &= \sqrt{2}\bar{\xi}_{\dot{\alpha}} \left\{ (\not{D}\psi)^{\dot{\alpha}} + \sqrt{2}g[\bar{\lambda}^{\dot{\alpha}}, \phi] \right\} \\
\delta F^{\dagger} &= \sqrt{2}\xi^{\alpha} \left\{ (\not{D}\bar{\psi})_{\alpha} + \sqrt{2}g[\phi^{\dagger}, \lambda_{\alpha}] \right\} \\
\delta A_{\mu} &= \bar{\lambda}\bar{\sigma}_{\mu}\xi + \bar{\xi}\bar{\sigma}_{\mu}\lambda \\
\delta\lambda &= -\bar{\sigma}_{\mu\nu}\xi F_{\mu\nu} + \xi D \\
\delta\bar{\lambda} &= -\bar{\xi}\sigma_{\mu\nu}F_{\mu\nu} + \bar{\xi}D \\
\delta D &= i\xi\not{D}\bar{\lambda} - i\bar{\xi}\not{D}\lambda
\end{aligned} \tag{C.28}$$

for the vector multiplets and

$$\begin{aligned}
\delta Q &= -\sqrt{2}i\xi q \\
\delta Q^{\dagger} &= \sqrt{2}i\bar{\xi}\bar{q} \\
\delta q &= -\sqrt{2}i\xi F_Q - \sqrt{2}\sigma_{\mu}\bar{\xi}D_{\mu}Q \\
\delta\bar{q} &= -\sqrt{2}\xi\sigma_{\mu}D_{\mu}Q^{\dagger} + \sqrt{2}i\bar{\xi}F_Q^{\dagger} \\
\delta F_Q &= \sqrt{2}\bar{\xi}\bar{\sigma}_{\mu}D_{\mu}q + 2g\bar{\xi}\bar{\lambda}Q \\
\delta F_Q^{\dagger} &= \sqrt{2}D_{\mu}\bar{q}\bar{\sigma}_{\mu}\xi + 2gQ^{\dagger}\xi\lambda \\
\delta\tilde{Q} &= -\sqrt{2}i\xi\tilde{q} \\
\delta\tilde{Q}^{\dagger} &= \sqrt{2}i\bar{\xi}\bar{\tilde{q}} \\
\delta\tilde{q} &= -\sqrt{2}i\xi F_{\tilde{Q}} - \sqrt{2}\sigma_{\mu}\bar{\xi}D_{\mu}\tilde{Q} \\
\delta\bar{\tilde{q}} &= -\sqrt{2}\xi\sigma_{\mu}D_{\mu}\tilde{Q}^{\dagger} + \sqrt{2}i\bar{\xi}F_{\tilde{Q}}^{\dagger} \\
\delta F_{\tilde{Q}} &= \sqrt{2}\bar{\xi}\bar{\sigma}_{\mu}D_{\mu}\tilde{q} - 2g\tilde{Q}\bar{\xi}\bar{\lambda} \\
\delta F_{\tilde{Q}}^{\dagger} &= \sqrt{2}D_{\mu}\tilde{q}\bar{\sigma}_{\mu}\xi - 2g\xi\lambda\tilde{Q}^{\dagger}
\end{aligned} \tag{C.29}$$

for the hypermultiplets. The other super transformation is given by $SU(2)_R$ rotation of this transformation.

D Regularization scheme

In this section, we discuss the regularization scheme. In section 5, we define the instanton measure so that the decoupling relation agrees with the \overline{DR} scheme[13]. In the \overline{DR} scheme, the dynamical scales of the low and high energy theories relate by the following decoupling relation:

$$\left(\frac{\Lambda_{0L}}{m}\right)^{b_{0L}} = \left(\frac{\Lambda_{0H}}{m}\right)^{b_{0H}} \text{ for any representation,} \tag{D.30}$$

where b_0 is the coefficient of the β function and m is the mass of the matter decoupled in the low energy theory. For $N = 2$ supersymmetric $SU(2)$ QCD, $b_0 = 4 - N_f$. We define the dynamical scale by $\Lambda^{b_0} = \mu^{b_0} e^{-8\pi^2/g^2(\mu)}$.

We can use the another scheme in the instanton calculus. For example, we will adopt a scheme, in which the decoupling relation is given by

$$\left(\frac{\alpha\Lambda_{0L}}{m}\right)^{b_{0L}} = \left(\frac{\alpha\Lambda_{0H}}{m}\right)^{b_{0H}} \text{ for adjoint representation,} \quad (\text{D.31})$$

$$\left(\frac{\Lambda_{0L}}{m}\right)^{b_{0L}} = \left(\frac{\Lambda_{0H}}{m}\right)^{b_{0H}} \text{ for fundamental representation,} \quad (\text{D.32})$$

where α is a numerical constant. In this scheme, the k -instanton measure of the $SU(N_c)$ vector multiplet is given by multiplying $\alpha^{2N_c k}$ to that of the \overline{DR} scheme. In this scheme, the result of the instanton calculus is the following:

$$u_1 = 2a^2 \times \begin{cases} \frac{\alpha^4}{2^3} \left(\frac{\Lambda_0}{a}\right)^4 & \text{for } N_f = 0, \\ 0 & \text{for } N_f \geq 1, \end{cases} \quad (\text{D.33})$$

$$u_2 = 2a^2 \times \begin{cases} \frac{5\alpha^8}{2^9} \left(\frac{\Lambda_0}{a}\right)^8 & \text{for } N_f = 0, \\ (-1)^{N_f} \frac{(2N_f - 1)\alpha^8}{2^{9-N_f} 3^{2N_f-3}} \left(\frac{\Lambda_{N_f}}{a}\right)^{8-2N_f} & \text{for } N_f \geq 1. \end{cases} \quad (\text{D.34})$$

The one-instanton calculus is consistent with the exact results Eq.(6.4), when we identify the dynamical scales as $\tilde{\Lambda}_0 = \sqrt{2}\alpha\Lambda_0$. By the decoupling relation for the fundamental representation, we obtain the relation between the dynamical scales,

$$\tilde{\Lambda}_{N_f}^{8-2N_f} = 16\alpha^8 \Lambda_{N_f}^{8-2N_f}. \quad (\text{D.35})$$

For any α the two-instanton calculus and the exact results agree for $N_f = 0, 1, 2$ and there appear the discrepancies for $N_f = 3, 4$, as we have already seen in section 6. For the supersymmetric $SU(3)$ Yang-Mills theory, the instanton calculus also agrees with the exact results in the one-instanton sector for any α . Therefore our result does not depend on the regularization scheme. The decoupling relations connecting the $N_f = 3$ and $N_f = 4$ theories are given by $\Lambda_3^2 = m_4^2 q$ for the instanton calculus and $\tilde{\Lambda}_3^2 = 64^2 m_4^2 q$ for the exact results. These relations are consistent with Eq.(D.35), if we choose $\alpha = 2$.

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